BOUNDEDNESS AND UNBOUNDEDNESS RESULTS FOR SOME MAXIMAL OPERATORS ON FUNCTIONS OF BOUNDED VARIATION

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ABSTRACT. We characterize the space $BV(I)$ of functions of bounded variation on an arbitrary interval $I \subset \mathbb{R}$, in terms of a uniform boundedness condition satisfied by the local uncentered maximal operator M_R from $BV(I)$ into the Sobolev space $W^{1,1}(I)$. By restriction, the corresponding characterization holds for $W^{1,1}(I)$. We also show that if U is open in $\mathbb{R}^d, d > 1$, then boundedness from $BV(U)$ into $W^{1,1}(U)$ fails for the local directional maximal operator M_T^v, the local strong maximal operator M_T^S, and the iterated local directional maximal operator $M_T^d \circ \cdots \circ M_T^1$. Nevertheless, if U satisfies a cone condition, then $M_T^S : BV(U) \to L^1(U)$ boundedly, and the same happens with M_T^v, $M_T^d \circ \cdots \circ M_T^1$, and M_R.

1. INTRODUCTION.

The *local* uncentered Hardy-Littlewood maximal operator M_R is defined in the same way as the uncentered Hardy-Littlewood maximal operator M, save for the fact that the supremum is taken over balls of diameter bounded by R, rather than all balls. The terms *restricted* and *truncated* have also been used in the literature to designate M_R. We showed in [AlPe] that if I is a bounded interval, then $M : BV(I) \to W^{1,1}(I)$ boundedly (Corollary 2.9). Here we complement this result by proving that for every interval I, including the case of infinite length, $M_R : BV(I) \to W^{1,1}(I)$ boundedly. Of course, no result of this kind can hold if we consider M instead of M_R, since $\|Mf\|_1 = \infty$ whenever f is nontrivial. We shall see that if $f \in BV(I)$, then $\|M_R f\|_{W^{1,1}(I)} \leq \max\{3(1 + 2\log^+ R), 4\}\|f\|_{BV(I)}$ (Theorem 2.7), and furthermore, the logarithmic order of growth of $c := \max\{3(1 + 2\log^+ R), 4\}$ cannot be improved (cf. Remark 2.8 below). Also, since c is nondecreasing in R, it provides a uniform bound for M_T whenever $T \leq R$. This observation leads to the following converse: Let $f \geq 0$. If there exists an $R > 0$ and a constant $c = c(f, R)$ such that for all $T \in (0, R]$, $M_T f \in W^{1,1}(I)$ and $\|M_T f\|_{W^{1,1}(I)} \leq c$, then $f \in BV(I)$. A fortiori, given a locally integrable $f \geq 0$, we have that $f \in BV(I)$ if and only if for every $R > 0$, $M_R f \in W^{1,1}(I)$ and there exists a constant $c = c(f, R)$ such that for all $T \in (0, R]$, $\|M_T f\|_{W^{1,1}(I)} \leq c$. By restriction to the functions f that are absolutely continuous on I, we obtain the corresponding characterization for $W^{1,1}(I)$. If f is real valued rather than nonnegative, since $f \in BV(I)$ (respectively $f \in W^{1,1}(I)$) if and only if both its positive and negative parts $f^+, f^- \in BV(I)$ (respectively $f^+, f^- \in W^{1,1}(I)$), we simply apply the previous criterion to $M_T f^+$ and $M_T f^-$.

It is natural to ask whether the uniform bound condition is necessary to ensure that $f \in BV(I)$, or whether it is sufficient just to require that for all $T \in \mathbb{R}$, $M_T f \in W^{1,1}(I)$. Uniform bounds are in fact needed (see Example 3.3).

In higher dimensions we show that boundedness fails for the local strong maximal operator (where the supremum is taken over rectangles with sides parallel to the axes and uniformly bounded diameters) and the local directional maximal operator (where the supremum is taken over uniformly bounded segments parallel to a fixed vector), cf. Theorem 2.21 below. But it is an open question whether the standard local maximal operator is bounded when $d > 1$, i.e., whether given a "sufficiently nice" open set $U \subset \mathbb{R}^d$, M_R maps $BV(U)$ boundedly into $W^{1,1}(U)$, or even into $BV(U)$. On the other hand, the direction from uniform boundedness of $M_T f^+$ and $M_T f^-$ to $f \in BV(U)$ follows immediately from the Lebesgue theorem on differentiation of integrals, even in the cases of the strong and directional maximal functions (cf. Theorem 3.1). All the maximal operators mentioned above map $BV(U)$ boundedly into $L^1(U)$, provided U satisfies a cone condition (Theorem 2.19), so the question of boundedness of M_R on $BV(U)$ is reduced to finding out how DM_R behaves.

Previous results on these topics include the following. In [Ha], Piotr Hajłasz utilized the local centered maximal operator to present a characterization, unrelated to the one given here, of the Sobolev space $W^{1,1}(\mathbb{R}^d)$. The boundedness of the centered Hardy-Littlewood maximal operator on the Sobolev spaces $W^{1,p}(\mathbb{R}^d)$, for $1 < p \le \infty$, was proven by Juha Kinnunen in [Ki]. A local version of this result, valid on $W^{1,p}(\Omega)$, $\Omega \subset \mathbb{R}^d$ open, appeared in [KiLi]. Additional work within this line of research includes the papers [HaOn], [KiSa], [Lu], [Bu], [Ko1], and [Ko2]. Of course, the case $p = 1$ is significantly different from the case $p > 1$. Nevertheless, in dimension $d = 1$, Hitoshi Tanaka showed (cf. [Ta]) that if $f \in W^{1,1}(\mathbb{R})$, then the uncentered maximal function Mf is differentiable a.e. and $\|DMf\|_1 \le 2\|Df\|_1$ (it is asked in [HaOn], Question 1, p. 169, whether an analogous result holds when $d > 1$). In [AlPe] we strengthened Tanaka's result, showing that if $f \in BV(I)$, then Mf is absolutely continuous and $\|DMf\|_1 \le |Df|(I)$, cf. [AlPe] Theorem 2.5.

Finally we mention that the local (centered and uncentered) maximal operator has been used in connection with inequalities involving derivatives, cf. [MaSh] and [AlPe]. Another instance of this type of application is given below (see Theorem 2.9).

2. DEFINITIONS, BOUNDEDNESS, AND UNBOUNDEDNESS RESULTS.

Let I be an interval and let λ (λ^d if $d > 1$) be Lebesgue measure. Since functions of bounded variation always have lateral limits, we can go from (a, b) to $[a, b]$ by extension, and viceversa by restriction. Thus, in what follows it does not matter whether I is open, closed or neither, nor whether it is bounded or has infinite length.

Definition 2.1. We say that $f : I \to \mathbb{R}$ is of bounded variation if its distributional derivative Df is a Radon measure with $|Df|(I) < \infty$, where $|Df|$ denotes the total variation of Df. In higher dimensions the definition is the same, save for the fact that Df is (co)vector valued rather than real valued. More precisely, if $U \subset \mathbb{R}^d$ is an open set and $f : U \to \mathbb{R}$

is of bounded variation, then Df is the vector valued Radon measure that satisfies, first, $\int_U f \operatorname{div} \phi dx = -\int_U \phi \cdot dDf$ for all $\phi \in C_c^1(U, \mathbb{R}^d)$, and second, $|Df|(U) < \infty$.

In addition to $|Df|(I) < \infty$, it is often required that $f \in L^1(I)$. We do so only when defining the space $BV(I)$, and likewise in higher dimensions. The next definition is given only for the one dimensional case, being entirely analogous when $d > 1$.

Definition 2.2. Given the interval I,

$$BV(I) := \{f : I \to \mathbb{R} | f \in L^1(I), Df \text{ is a Radon measure, and } |Df|(I) < \infty\},$$

and

$$W^{1,1}(I) := \{f : I \to \mathbb{R} | f \in L^1(I), Df \text{ is a function, and } Df \in L^1(I)\}.$$

It is obvious that $W^{1,1}(I) \subset BV(I)$ properly. The Banach space $BV(I)$ is endowed with the norm $\|f\|_{BV(I)} := \|f\|_1 + |Df|(I)$, and $W^{1,1}(I)$, with the restriction of the BV norm, i.e., $\|f\|_{W^{1,1}(I)} := \|f\|_1 + \|Df\|_1$.

Definition 2.3. The canonical representative of f is the function

$$\overline{f}(x) := \limsup_{\lambda(I) \to 0, x \in I} \frac{1}{\lambda(I)} \int_I f(y) dy.$$

In dimension $d = 1$, bounded variation admits an elementary, equivalent definition. Given $P = \{x_1, \ldots, x_L\} \subset I$ with $x_1 < \cdots < x_L$, the variation of the *function* $f : I \to \mathbb{R}$ associated to the partition P is defined as $V(f, I, P) := \sum_{j=2}^L |f(x_j) - f(x_{j-1})|$, and the variation of f on I, as $V(f, I) := \sup_P V(f, I, P)$, where the supremum is taken over all partitions P of I. Then f is of bounded variation if $V(f, I) < \infty$. As it stands this definition is not L^p compatible, in the sense that modifying f on a set of measure zero can change $V(f, I)$, and even make $V(f, I) = \infty$. To remove this defect one simply says that f is of bounded variation if $V(\overline{f}, I) < \infty$. It is then well known that $|Df|(I) = V(\overline{f}, I)$.

Definition 2.4. Let $f : I \to \mathbb{R}$ be measurable and finite a.e.. The non-increasing rearrangement f^* of f is defined for $0 < t < \lambda(I)$ as

$$f^*(t) = \sup_{\lambda(E)=t} \inf_{y \in E} |f(y)|.$$

The function f^* is non-increasing and equimeasurable with $|f|$. Furthermore,

(2.4.1)
$$\int_I f(y) dy = \int_0^{\lambda(I)} f^*(t) dt.$$

For these and other basic properties of rearrangements see [BeSh, Chapter 2]. We mention that the same definition can be used for general measure spaces.

In the next definition, $\operatorname{diam}(A)$ denotes the diameter of a set A, $U \subset \mathbb{R}^d$ denotes an open set, and $B \subset \mathbb{R}^d$ a ball with respect to some fixed norm.

Definition 2.5. Given a locally integrable function $f : U \to \mathbb{R}$, the *local* uncentered Hardy-Littlewood maximal function $M_R f$ is defined by

$$M_R f(x) := \sup_{x \in B \subset U, \mathrm{diam}\, B \leq R} \frac{1}{\lambda^d(B)} \int_B |f(y)|dy.$$

Of course, if the bound R is eliminated then we get the usual uncentered Hardy-Littlewood maximal function Mf.

As noted in the introduction, the terms *restricted* and *truncated* have also been used in the literature to designate M_R, but we prefer *local* for the reasons detailed in Remark 2.4 of [AlPe]. Next we recall the well known weak type (1,1) inequality satisfied by M in dimension 1, with the sharp constant 2. For all $f \in L^1(I)$ and all $t > 0$,

$$(2.5.1) \qquad\qquad (Mf)^*(t) \leq 2\|f\|_1/t.$$

Definition 2.6. Let $U \subset \mathbb{R}^d$ be an open set, and let $f : U \to \mathbb{R}$ be a locally integrable function. By a rectangle R we mean a rectangle with sides parallel to the axes. The local uncentered *strong* Hardy-Littlewood maximal function $M_T^S f$ is defined by

$$M_T^S f(x) := \sup_{x \in R \subset U, \mathrm{diam}(R) \leq T} \frac{1}{\lambda^d(R)} \int_R |f(y)|dy.$$

Next, let $v \in \mathbb{R}$ be a fixed vector, and let J denote a (one dimensional) segment in $\mathbb{R}^d$ parallel to v. The local uncentered *directional* Hardy-Littlewood maximal function $M_T^v f$ is defined by

$$M_T^v f(x) := \sup_{x \in J \subset U, \lambda(J) \leq T} \frac{1}{\lambda(J)} \int_J |f(y)|dy.$$

If $v = e_i$, then we write M_T^i instead of $M_T^{e_i}$.

We shall also be interested in the composition $M_T^d \circ \cdots \circ M_T^1$ of the d local directional maximal operators in the directions of the coordinate axes, since such composition controls M_T^S pointwise. But first, we deal with the one dimensional case.

Theorem 2.7. *If $|f| \in BV(I)$, then $M_R f \in W^{1,1}(I)$ and furthermore, $\|M_R f\|_{W^{1,1}(I)} \leq 3(1+2\log^+ R)\|f\|_{L^1(I)} + 4\,|D|f||\,(I)$. Hence, $\|M_R f\|_{W^{1,1}(I)} \leq \max\{3(1+2\log^+ R), 4\}\|f\|_{BV(I)}$.*

Proof. Note that for any interval J and any $h \in BV(J)$

$$(2.7.1) \qquad \|h\|_{L^\infty(J)} \leq \mathrm{essinf}\, |h| + |Dh|(J) \leq \frac{\|h\|_{L^1(J)}}{\lambda(J)} + |Dh|(J).$$

Now, given $f : I \to \mathbb{R}$, if $|D|f||$ is a finite Radon measure on I, then $M_R f$ is absolutely continuous on I and $\|DM_R f\|_{L^1(I)} \leq |D|f||\,(I)$ by [AlPe], Theorem 2.5 (we mention that for this bound on the size of the derivative, the hypothesis $f \in L^1(I)$ is not needed). Thus, it is enough to prove that given $|f| \in BV(I)$,

$$(2.7.2) \qquad \|M_R f\|_{L^1(I)} \leq 3(1 + 2\log^+ R)\|f\|_{L^1(I)} + 3|D|f||(I).$$

We may assume that $0 \le f = \bar{f}$, since this does not change any value of $M_R f$. Given $k \in \mathbb{Z}$ we denote by I_k and J_k the (possibly empty) intervals $I \cap [kR, (k+1)R)$ and $I \cap [(k-1)R, (k+2)R)$ respectively. We also set $f_k := f|_{J_k}$. Fix k. Then

$$(2.7.3) \qquad \int_{I_k} M_R f(x) dx = \int_{I_k} M_R f_k(x) dx \le \int_{I_k} M f_k(x) dx.$$

Suppose first that $\lambda(I_k) \le 1$. From (2.7.1) we get

$$(2.7.4) \qquad \int_{I_k} M f_k(x) dx \le \lambda(I_k) \|f_k\|_{L^\infty(J_k)} \le \|f_k\|_{L^1(J_k)} + |Df_k|(J_k).$$

And if $\lambda(I_k) > 1$, then from (2.4.1) and (2.5.1) we obtain

$$(2.7.5) \qquad \int_{I_k} M f_k(x) dx = \int_0^{\lambda(I_k)} (M f_k)^*(t) dt = \int_0^1 + \int_1^{\lambda(I_k)}$$

$$\le \|f_k\|_{L^\infty(J_k)} + 2\|f_k\|_{L^1(J_k)} \int_1^{\lambda(I_k)} t^{-1} dt$$

$$\le (1 + 2\log R)\|f_k\|_{L^1(J_k)} + |Df_k|(J_k).$$

Since the intervals I_k are all disjoint, and each nonempty I_k is contained in J_{k-1}, J_k and J_{k+1}, having empty intersection with all the other J_i's, the estimates (2.7.4) and (2.7.5) yield

$$(2.7.6) \qquad \|M_R f\|_{L^1(I)} = \sum_{-\infty}^{\infty} \int_{I_k} M_R f(x) dx$$

$$\le \sum_{-\infty}^{\infty} \left((1 + 2\log^+ R)\|f_k\|_{L^1(J_k)} + |Df_k|(J_k) \right)$$

$$= 3 \sum_{-\infty}^{\infty} (1 + 2\log^+ R)\|f_k\|_{L^1(I_k)} + 3 \sum_{-\infty}^{\infty} |Df_k|(I_k)$$

$$= 3(1 + 2\log^+ R)\|f\|_{L^1(I)} + 3|Df|(I).$$

Thus,

$$\|M_R f\|_{BV(I)} \le 3(1 + 2\log^+ R)\|f\|_{L^1(I)} + 4|Df|(I) \le \max\{3(1 + 2\log^+ R), 4\}\|f\|_{BV(I)}.$$

$\square$

Remark 2.8. The example $f : \mathbb{R} \to \mathbb{R}$ given by $f := \chi_{[0,1]}$ shows that the logarithmic order of growth in the preceding theorem is the correct one. Here all the relevant quantities can be easily computed: $\|f\|_{L^1(\mathbb{R})} = 1$, $|Df|(\mathbb{R}) = 2$, $\|M_R f\|_{L^1(\mathbb{R})} = 1 + 1/R + 2\log R$ for $R \ge 1$, and $|DM_R f|(\mathbb{R}) = 2$ (for all $R > 0$).

As noted in [AlPe], this kind of bounds on the size of maximal functions and their derivatives can be used to obtain variants of the classical Poincaré inequality, as well as other inequalities involving derivatives, under less regularity, by using $DM_R f$ (a function) instead of Df (a Radon measure). Here we present another instance of the same idea, a Poincaré type inequality involving $\|M_R f\|_1$; the argument is standard but short, so we include it for the reader's convenience.

Given a compactly supported function f, denote by $N(f, R) := \operatorname{supp} f + [-R, R] \subset \mathbb{R}$ the closed R-neighborhood of its support, that is, the set of all points at distance less than or equal to R from the support of f.

Theorem 2.9. *Let* $f \in BV(\mathbb{R})$ *be compactly supported. Then for all* $R > 0$, *we have* $\|f\|_2^2$

$$\leq \min\left\{ \frac{(3(1 + 2\log^+ R))^2}{\lambda(N(f, R))}\|f\|_{BV(\mathbb{R})}^2 + \left(\frac{(\lambda(N(f, R)))^2}{2}\right)\|DM_R f\|_2^2, \lambda(N(f, R))^2\|DM_R f\|_2^2, \right\}.$$

Proof. Let $x < y$ be points in $\mathbb{R}$. By the Fundamental Theorem of Calculus,

$$M_R f(y) - M_R f(x) = \int_x^y DM_R f(t)dt \leq \|DM_R f\|_1.$$

Squaring and integrating with respect to x and y over $N(f, R)^2$, we get

$$\|M_R f\|_2^2 \leq \frac{\|M_R f\|_1^2}{\lambda(N(f, R))} + \|DM_R f\|_1^2 \left(\frac{\lambda(N(f, R))}{2}\right).$$

Since $\|f\|_2^2 \leq \|M_R f\|_2^2$, using (2.7.6) and either Jensen or Hölder inequality we obtain

$$\|f\|_2^2 \leq \frac{(3(1 + 2\log^+ R))^2}{\lambda(N(f, R))}\|f\|_{BV(\mathbb{R})}^2 + \left(\frac{(\lambda(N(f, R)))^2}{2}\right)\|DM_R f\|_2^2.$$

On the other hand, integrating $M_R f(y) = \int_\infty^y DM_R f(t)dt \leq \|DM_R f\|_1$ and repeating the previous steps we get

$$\|f\|_2^2 \leq \lambda(N(f, R))^2\|DM_R f\|_2^2.$$

$\square$

Remark 2.10. In connection with the preceding inequality, we point out that if $1 < p < \infty$ and $f \in W^{1,p}(\mathbb{R})$, then $\|DM_R f\|_p \leq c_p\|Df\|_p$, with c_p independent of R. Of course, the interest of the result lies in the fact that we can have $\|DM_R f\|_p < \infty$ even if Df is not a function (standard example, $f = \chi_{[0,1]}$). The cases $p = 1, \infty$ are handled in [AlPe], Theorems 2.5 and 5.6. There we have $\|DM_R f\|_p \leq \|Df\|_p$. To see why $\|DM_R f\|_p \leq c_p\|Df\|_p$ holds with c_p independent of R, repeat the sublinearity argument from [Ki], Remark 2.2 (i) (cf. also [HaOn], Theorem 1) using $M_R f \leq Mf$ to remove the dependency of the constant on R.

We shall consider next the local strong, directional, and iterated directional maximal operators, proving boundedness from $BV(U)$ into $L^1(U)$ and lack of boundedness from $BV(U)$ into $BV(U)$. Of course, since the strong maximal operator dominates pointwise (up to a

constant factor) the maximal operator associated to an arbitrary norm, we also obtain the boundedness of M_R from $BV(U)$ into $L^1(U)$.

Remark 2.11. It is possible to define $BV(U)$, where U is open in $\mathbb{R}^d$, without knowing a priori that $|Df|$ is a Radon measure: Write

$$(2.11.1) \qquad \int_U |Df| := \sup \left\{ \int_U f \operatorname{div} g : g \in C_c^1(U, \mathbb{R}^d), \|g\|_\infty \leq 1 \right\}.$$

Then $f \in BV(U)$ if $f \in L^1(U)$ and $\int_U |Df| < \infty$ (cf., for instance, Definition 1.3, pg. 4 of [Giu], or Definition 3.4, pg. 119 and Proposition 3.6, pg. 120 of [AFP]). Integration by parts immediately yields that if $f \in C^1(U)$, then

$$\int_U |Df| = \int_U |\nabla f| dx,$$

(this is Example 1.2 of [Giu]). With this approach one has the following semicontinuity and approximation results (cf. Theorems 1.9 and 1.17 of [Giu]), without any reference to Radon measures.

Theorem 2.12. *If a sequence of functions $\{f_n\}$ in $BV(U)$ converges in $L^1_{loc}(U)$ to f, then $\int_U |Df| \leq \liminf_n \int_U |Df_n|$.*

Theorem 2.13. *If $f \in BV(U)$, then there exists a sequence of functions $\{f_n\}$ in $BV(U) \cap C^\infty(U)$ such that $\lim_n \int_U |f - f_n| dx = 0$ and $\int_U |Df| = \lim_n \int_U |Df_n|$.*

Note that by passing to a subsequence, we may also assume that $\{f_n\}$ converges to f almost everywhere.

If one uses the definition of $BV(U)$ given in Remark 2.11, the fact that Df is a Radon measure is obtained a posteriori via the Riesz Representation Theorem. Then of course $\int_U |Df| = |Df|(U)$.

Definition 2.14. A finite cone C of height r, vertex at 0, axis v, and aperture angle α, is the subset of $B(0, r)$ consisting of all vectors y such that the angle between y and v is less than or equal to $\alpha/2$. A finite cone C_x with vertex at x, is a set of the form $x + C$, where the vertex of C is 0. Finally, an open set U satisfies a cone condition if there exists a fixed finite cone C such that every $x \in U$ is the vertex of a cone obtained from C by a rigid motion.

We shall assume a cone condition in order to have available the following special case of the Sobolev embedding theorem (see, for instance, Theorem 4.12, pg. 85 of [AdFo]). Of course, other type of conditions which also ensure the existence of such an embedding could be used instead (e.g., U is an extension domain). The next Theorem and its Corollary are well known and included here for the sake of readability.

Theorem 2.15. *Let the open set $U \subset \mathbb{R}^d$ satisfy a cone condition. Then there exists a constant $c > 0$, depending only on U, such that for all $f \in W^{1,1}(U)$, $\|f\|_{L^{\frac{d}{d-1}}(U)} \leq c\|f\|_{W^{1,1}(U)}$.*

Corollary 2.16. *Let the open set $U \subset \mathbb{R}^d$ satisfy a cone condition. Then there exists a constant $c > 0$, depending only on U, such that for all $f \in BV(U)$, $\|f\|_{L^{\frac{d}{d-1}}(U)} \le c\|f\|_{BV(U)}$.*

Proof. Let $\{f_n\}$ be a sequence of functions in $BV(U) \cap C^\infty(U)$ such that $f_n \to f$ a.e., $\lim_n \int_U |f - f_n| dx = 0$, and $\int_U |Df| = \lim_n \int_U |\nabla f_n| dx$. By Fatou's lemma and Theorem 2.15, $\|f\|_{L^{\frac{d}{d-1}}(U)} \le \liminf_n \|f_n\|_{L^{\frac{d}{d-1}}(U)} \le \lim_n c\|f_n\|_{W^{1,1}(U)} = c\|f\|_{BV(U)}$. $\qquad\square$

The next definition and lemma are valid for an arbitrary set $E \subset \mathbb{R}^k$, with measure defined by the restriction of the Lebesgue outer measure to the σ-algebra of all intersections of Lebesgue sets with E.

Definition 2.17. Let $E \subset \mathbb{R}^k$ and $r \ge 1$. A function g belongs to the Banach space $L(\log^+ L)^r(E)$ if for some $t > 0$ we have

$$(2.17.1) \qquad \int \frac{|g(x)|}{t} \left(\log^+ \frac{|g(x)|}{t} \right)^r dx < \infty.$$

In that case the Luxemburg norm of g is

$$\|g\|_{L(\log^+ L)^r} := \inf \left\{ t > 0 : \int \frac{|g(x)|}{t} \left(\log^+ \frac{|g(x)|}{t} \right)^r dx \le 1 \right\}.$$

Note that by monotone convergence the inequality

$$\int \frac{|g(x)|}{t} \left(\log^+ \frac{|g(x)|}{t} \right)^r dx \le 1$$

holds when $t = \|g\|_{L(\log^+ L)^r}$.

We mention that on finite measure spaces, the condition of Definition 2.17 is equivalent to the seemingly stronger requirement that for all $t > 0$, (2.17.1) hold.

The next lemma must be well known, but we include it for the reader's convenience. While stated for all $r \ge 1$, we only need the cases $r = 1$ (used in Remark 2.20), $r = d - 1$ (used in Theorem 3.1) and $r = d$ (used in Theorem 2.19).

Lemma 2.18. *Let $E \subset \mathbb{R}^d$, where $d \ge 2$, and let $r \ge 1$. If $g \in L^{\frac{d}{d-1}}(E)$, then $g \in L(\log^+ L)^r(E)$ and $\|g\|_{L(\log^+ L)^r(E)} \le (r(d-1))^{\frac{r(d-1)}{d}} \|g\|_{L^{\frac{d}{d-1}}(E)}$.*

Proof. Note that $\log^+ y \le y^\alpha/\alpha$ for all $y, \alpha > 0$, so given $t > 0$, if we set $y = \frac{|g(x)|}{t}$ and $\alpha = \frac{1}{r(d-1)}$, we get

$$\int \frac{|g(x)|}{t} \left(\log^+ \frac{|g(x)|}{t} \right)^r dx \le (r(d-1))^r \left\| \frac{g}{t} \right\|_{L^{\frac{d}{d-1}}(E)}^{\frac{d}{d-1}}.$$

Now let $t_0 < \|g\|_{L(\log^+ L)^r}$. Then $1 < (r(d-1))^r \left\| \frac{g}{t_0} \right\|_{L^{\frac{d}{d-1}}(E)}^{\frac{d}{d-1}}$, from which it follows that

$$\|g\|_{L(\log^+ L)^r(E)} \le (r(d-1))^{\frac{r(d-1)}{d}} \|g\|_{L^{\frac{d}{d-1}}(E)}. \qquad\square$$

The proof of the next result is similar to that of Theorem 2.7. We indicate the main differences: 1) In Theorem 2.7, since $d = 1$, no cone condition appears and we give a fully explicit constant; 2) when $d = 1$, we use the trivial embedding of $BV(I)$ in L^∞ given in (2.7.1) instead of Corollary 2.16 and Lemma 2.18; 3) for $d > 1$, bounds on the distributional gradient of the corresponding maximal operator are either false or not known.

Theorem 2.19. *Let the open set $U \subset \mathbb{R}^d$ satisfy a cone condition. For every $R > 0$, the local iterated directional maximal operator $M_R^d \circ \cdots \circ M_R^1$ and the local strong maximal operator M_R^S map $BV(U)$ into $L^1(U)$ boundedly. Hence, so do the following operators: The standard local uncentered maximal operator M_R associated to an arbitrary norm, the local directional maximal operator M_R^v, and $M_R^{i_k} \circ \cdots \circ M_R^{i_1}$, where $1 \le k < d$ and $i_1 < \cdots < i_k$. In fact, if S_R is any of the above maximal operators, then there exists a constant $c > 0$, which depends only on the open set U, such that for all $f \in BV(U)$,*

$$(2.19.1) \qquad \|S_R f\|_{L^1(U)} \le c \left(\|f\|_{BV(U)} + (\log^+ R)^d \|f\|_{L^1(U)} \right).$$

Proof. By Corollary 2.16, it is enough to show that

$$(2.19.2) \qquad \|S_R f\|_{L^1(U)} \le c \left(\|f\|_{L^{d/(d-1)}(U)} + (\log^+ R)^d \|f\|_{L^1(U)} \right).$$

Now we can assume that $U = \mathbb{R}^d$. Else, we extend f without changing the right hand side of (2.19.2), by setting $f = 0$ on $\mathbb{R}^d \setminus U$.

The reason we are interested in having $U = \mathbb{R}^d$ is that later on, we will use the pointwise equivalence on $\mathbb{R}^d$ of maximal functions associated to different norms.

By η we denote a generic d-tuple of integers $(n_1, \ldots, n_d) \in \mathbb{Z}^d$. For $\eta \in \mathbb{Z}^d$, we define the cubes $I_\eta = [n_1 R, (n_1 + 1)R) \times \cdots \times [n_d R, (n_d + 1)R)$ and $J_\eta = [(n_1 - 1)R, (n_1 + 2)R) \times \cdots \times [(n_d - 1)R, (n_d + 2)R)$. Set $f_\eta = f|_{J_\eta}$.

We want to estimate

$$\alpha_\eta := \int_{I_\eta} M_R^d \circ \cdots \circ M_R^1 f(x) dx$$

$$= \int_{I_\eta} M_R^d \circ \cdots \circ M_R^1 f_\eta(x) dx \le \int_{I_\eta} M^d \circ \cdots \circ M^1 f_\eta(x) dx.$$

From [Fa, §I. Theorem 1], we get

$$(2.19.3) \qquad \lambda^d(\{M^d \circ \cdots \circ M^1 f_\eta > 4t\}) \le C \int_{J_\eta} \frac{|f_\eta(x)|}{t} \left(\log^+ \frac{|f_\eta(x)|}{t} \right)^{d-1} dx,$$

where C is a constant that depends only on d. Moreover, calling $A = \|f_\eta\|_{L(\log^+ L)^d}$ and using (2.19.3) we obtain

$$\alpha_\eta = 4 \int_0^\infty \lambda^d(I_\eta \cap \{M_R^d \circ \cdots \circ M_R^1 f_\eta(x) > 4t\}) dt = 4 \int_0^{A/R^d} + 4 \int_{A/R^d}^\infty$$

$$(2.19.4) \qquad \le 4A + 4C \int_{A/R^d}^\infty \int_{J_\eta} \frac{|f_\eta(x)|}{t} \left(\log^+ \frac{|f_\eta(x)|}{t} \right)^{d-1} dx dt = 4A + B.$$

Let $\tilde{J}_\eta := J_\eta \cap \{|f(x)| > A/R^d\}$. Applying the Fubini-Tonelli Theorem and the change of variable $y(t) = \log \frac{|f_\eta(x)|}{t}$ we have

$$B = 4C \int_{\tilde{J}_\eta} \int_{A/R^d}^{|f_\eta(x)|} \frac{|f_\eta(x)|}{t} \left(\log \frac{|f_\eta(x)|}{t} \right)^{d-1} dtdx$$

$$= 4C \int_{\tilde{J}_\eta} |f_\eta(x)| dx \int_0^{\log^+ \frac{|f_\eta(x)|R^d}{A}} y^{d-1} dy$$

$$= \frac{4C}{d} \int_{\tilde{J}_\eta} |f_\eta(x)| \left(\log \frac{|f_\eta(x)|}{A} + d \log R \right)^d dx$$

$$\leq \frac{4C2^d}{d} \int_{J_\eta} |f_\eta(x)| \left(\left(\log^+ \frac{|f_\eta(x)|}{A} \right)^d + d^d \left(\log^+ R \right)^d \right) dx$$

$$= \frac{4C2^d}{d} \left(A \int_{J_\eta} \frac{|f_\eta(x)|}{A} \left(\log^+ \frac{|f_\eta(x)|}{A} \right)^d dx + d^d \|f_\eta\|_{L^1(J_\eta)} (\log^+ R)^d \right)$$

$$(2.19.5) \qquad\qquad \leq \frac{4C2^d}{d} \left(A + d^d \|f_\eta\|_{L^1(J_\eta)} (\log^+ R)^d \right).$$

Putting together (2.19.4), (2.19.5), and Lemma 2.18, we get

$$\alpha_\eta \leq C' \left(\|f_\eta\|_{L^{d/(d-1)}(J_\eta)} + \|f_\eta\|_{L^1(J_\eta)} (\log^+ R)^d \right).$$

Next we sum over all d-tuples $\eta \in \mathbb{Z}^d$. Since a point in $\mathbb{R}^d$ cannot be contained in more than 3^d different cubes of type J, we conclude that for some $c > 0$,

$$(2.19.6) \qquad \int_{\mathbb{R}^d} M_R^d \circ \cdots \circ M_R^1 f(x) dx \leq c \left(\|f\|_{L^{d/(d-1)}(\mathbb{R}^d)} + \|f\|_{L^1(\mathbb{R}^d)} (\log^+ R)^d \right).$$

Since $M_R^S f(x) \leq M_R^d \circ \cdots \circ M_R^1 f(x)$ for almost all $x \in \mathbb{R}^d$, the same inequality holds for $M_R^S f$. Likewise, M_R^S dominates pointwise the maximal operator M_R associated to the l^∞ norm (i.e., to cubes), so (2.19.1) also holds for M_R. Since local maximal operators associated to different norms are pointwise comparable by the equivalence of all norms in $\mathbb{R}^d$, inequality (2.19.1) holds, perhaps with a different value of c, for the maximal operator M_R defined by any given norm. Finally, if $1 \leq k < d$ and $i_1 < i_2 < \cdots < i_k$, we have $M_R^{i_k} \circ \cdots \circ M_R^{i_1} f(x) \leq M_R^d \circ \cdots \circ M_R^1 f(x)$ for all $x \in \mathbb{R}^d$, and M_R^v obviously satisfies the same bounds as M_R^1, so (2.19.1) holds for all the operators under consideration. $\qquad\square$

Remark 2.20. It is possible to obtain bounds for M_R directly, using essentially the same proof as in the previous theorem, rather than deriving them from the corresponding bounds for M_R^S. In fact, a direct approach yields a lower order of growth, $O(\log R)$ instead of $O((\log R)^d)$. More precisely, replace in the proof $L(\log^+ L)^d$ by $L(\log^+ L)$, and inequality (2.19.3) by the

following well known refinement (due to N. Wiener, cf. [Wi, Theorem 4′]) of the weak type inequality:

$$\lambda^d(\{Mf > t\}) \leq \frac{C}{t} \int_{\{|f|>t/2\}} |f(x)|dx \qquad \text{for all } t > 0.$$

Then argue as before, to get

$$\int_U M_R f(x)dx \leq c\left(\|f\|_{BV(U)} + \|f\|_{L^1(U)} \log^+ R\right).$$

An analogous remark can be made with respect to the operators $M_R^{i_k} \circ \cdots \circ M_R^{i_1}$ and M_R^v, obtaining orders of growth $O(\log^k R)$ and $O(\log R)$ respectively.

Theorem 2.21. *Let $d > 1$ and let $U \subset \mathbb{R}^d$ be open. Given any $R > 0$, the following maximal operators are unbounded on $BV(U)$: The local directional maximal operator M_R^v, the local iterated directional maximal operator $M_R^d \circ \cdots \circ M_R^1$, and the local strong maximal operator M_R^S.*

Proof. We will show that if S_R denotes any of the maximal operators considered in the statement of the theorem, then there exists a sequence of characteristic functions $f_{1/n}$ such that $\lim_{n\to\infty} \|f_{1/n}\|_{BV(U)} = 0$ and

$$\lim_{n\to\infty} \frac{|DS_R(f_{1/n})|(U)}{\|f_{1/n}\|_{BV(U)}} = \infty.$$

In fact, the same result holds for the corresponding nonlocal maximal operators, which can be included in the notation S_R by allowing the possibility $R = \infty$, as we do in this proof. So we take $0 < R \leq \infty$. Actually it is enough to consider $2 < R \leq \infty$, since the argument we give below adapts to smaller values for R just by rescaling. Similarly it is enough to consider the case $U = \mathbb{R}^d$. We start with M_R^v. By a rotation we may assume that $v = e_1$. For notational simplicity, we will write the proof for the case $d = 2$ only. Fix R. Given $0 < \delta < 1$, set $f_\delta(x) := \chi_{[0,\delta]^2}(x)$. Then

$$\|f_\delta\|_1 = \delta^2$$

and, since $|Df_\delta|(\mathbb{R}^2)$ is just the perimeter of the square $[0,\delta]^2$ (cf., for instance, Exercise 3.10 pg. 209 of [AFP]),

$$|Df_\delta|(\mathbb{R}^2) = 4\delta.$$

Thus

(2.21.1) $$\|f_\delta\|_{BV(\mathbb{R}^2)} = O(\delta) \text{ when } \delta \to 0.$$

Next, let $\delta \leq x \leq 1$, and $0 \leq y \leq \delta$. It is then easy to check that

$$M_R^1(f_\delta)(x,y) = \frac{\delta}{x}.$$

Given $\delta \leq t < 1$, the level sets $E_t := \{M_R^1(f_\delta) > t\}$ are rectangles, with perimeter

$$|D\chi_{E_t}|(\mathbb{R}^2) \geq 2\delta + \frac{2\delta}{t}.$$

By the coarea formula for BV functions (cf. Theorem 3.40, pg. 145 of [AFP]), we have
(2.21.2)

$$|DM_R^1 f_\delta|(\mathbb{R}^2) = \int_{-\infty}^{\infty} |D\chi_{E_t}|(\mathbb{R}^2)dt \geq \int_\delta^1 |D\chi_{E_t}|(\mathbb{R}^2)dt \geq 2\delta \int_\delta^1 \left(1 + \frac{1}{t}\right) dt = \Theta\left(\delta \log \frac{1}{\delta}\right).$$

where Θ stands for the exact order of growth. From (2.21.1) and (2.21.2) we obtain

$$(2.21.3) \qquad \frac{|DM_R^1(f_\delta)|(\mathbb{R}^2)}{\|f_\delta\|_{BV((\mathbb{R}^2)}} \to \infty \text{ when } \delta \to 0,$$

as was to be proven.

Note next that on $[0,1] \times [0,\delta]$ the three maximal functions $M_R^1 f_\delta$, $M_R^2 \circ M_R^1 f_\delta$ and $M_R^S f_\delta$ take the same values, from which it easily follows that for $\delta \leq t < 1$,

$$|D\chi_{\{M_R^2 \circ M_R^1(f_\delta) > t\}}|(\mathbb{R}^2) \geq 2\delta + \frac{2\delta}{t}$$

and

$$|D\chi_{\{M_R^S(f_\delta) > t\}}|(\mathbb{R}^2) \geq 2\delta + \frac{2\delta}{t}.$$

Thus, the analogous statement to (2.21.3) holds for $M_R^2 \circ M_R^1 f_\delta$ and $M_R^S f_\delta$ also. □

A standard mollification argument shows that the preceding maximal operators are not bounded on $W^{1,1}(U)$ either.

3. Converses and a one dimensional characterization.

Recall that f^+ and f^- denote respectively the positive and negative parts of f. Now, for any open set $U \subset \mathbb{R}^d$, $f \in BV(U)$ if and only if both $f^+ \in BV(U)$ and $f^- \in BV(U)$. This can be seen as follows: If $f \in BV(U)$, it is immediate from the definition 2.11.1 contained in Remark 2.11 that $\int_U |Df| \geq \int_U |D(f^+)|$ and $\int_U |Df| \geq \int_U |D(f^-)|$, so $f^+, f^- \in BV(U)$. On the other hand, if both $f^+, f^- \in BV(U)$, then there are sequences $\{g_n\}$ and $\{h_n\}$ of C^∞ functions that approximate f^+ and f^- respectively, in the sense of Theorem 2.13. Since $g_n - h_n \to f$ in $L^1(U)$, by semicontinuity $|Df|(U) \leq \liminf_n \int_U |\nabla(g_n - h_n)|dx \leq \liminf_n \int_U |\nabla g_n|dx + \liminf_n \int_U |\nabla h_n|dx = |D(f^+)|(U) + |D(f^-)(U)|$. Hence $f \in BV(U)$.

Theorem 3.1. *Let $U \subset \mathbb{R}^d$ be an open set and let $f : U \to \mathbb{R}$ be locally integrable. Suppose that there exists a sequence $\{a_n\}_1^\infty$ with $\lim_n a_n = 0$ and a constant c such that for all n, $M_{a_n} f^+ \in W^{1,1}(U)$, $M_{a_n} f^- \in W^{1,1}(U)$, $\|M_{a_n} f^+\|_{W^{1,1}(U)} \leq c$, and $\|M_{a_n} f^-\|_{W^{1,1}(U)} \leq c$. Then $f \in BV(U)$. The same happens if instead of M_R we consider either the local directional maximal operator, or, under the additional hypothesis that U satisfies a cone condition, the local strong maximal operator.*

Proof. Consider first f^+. By the Lebesgue Theorem on differentiation of integrals we have that $\lim_n M_{a_n} f^+ = f^+$ a.e., so by dominated convergence, $M_{a_n} f^+ \to f^+$ in $L^1(U)$, and by Theorem 2.12, $\int_U |Df^+| \leq \liminf_n \int_U |DM_{a_n} f^+| \leq c < \infty$. Repeating the argument for f^- we get $|Df|(U) \leq |Df^+|(U) + |Df^-|(U) < \infty$. The result for the local strong maximal

operator follows from the well known Theorem of Jessen, Marcinkiewicz and Zygmund ([JMZ]) stating that basis of rectangles (with sides parallel to the axes) differentiates $L(\log^+ L)_{loc}^{d-1}(U)$, and hence $BV(U)$ (cf. Corollary 2.16 and Lemma 2.18; for the first embedding we use the cone condition). Finally, the weak type $(1,1)$ boundedness of M_T^v (which is obtained from the one dimensional result and the Fubini-Tonelli Theorem) also entails, by the standard argument, the corresponding differentiation of integrals result, so $\lim_n M_{a_n}^v f^+ = f^+$ and $\lim_n M_{a_n}^v f^- = f^-$. $\qquad\square$

For intervals $I \subset \mathbb{R}$ we have the following characterization.

Theorem 3.2. *Let $f : I \to \mathbb{R}$ be locally integrable. Then the following are equivalent:*
   a) $f \in BV(I)$.
   b) $M_R f^+ \in W^{1,1}(I)$, $M_R f^- \in W^{1,1}(I)$, $\|M_R f^+\|_{W^{1,1}(I)} \le 3(1 + 2\log^+(R))\|f^+\|_{L^1(I)} + 4|Df^+|(I)$, and $\|M_R f^-\|_{W^{1,1}(I)} \le 3(1 + 2\log^+(R))\|f^-\|_{L^1(I)} + 4|Df^-|(I)$.
   c) There exists a sequence $\{a_n\}_1^\infty$ with $\lim_n a_n = 0$ and a constant $c = c(f, \{a_n\}_1^\infty)$ such that for all n, $M_{a_n} f^+ \in W^{1,1}(I)$, $M_{a_n} f^- \in W^{1,1}(I)$, $\|M_{a_n} f^+\|_{W^{1,1}(I)} \le c$, and $\|M_{a_n} f^-\|_{W^{1,1}(I)} \le c$.
   d) There exists an $R > 0$ and a constant $c = c(f, R)$ such that for all $T \in (0, R]$, $M_T f^+ \in W^{1,1}(I)$, $M_T f^- \in W^{1,1}(I)$, $\|M_T f^+\|_{W^{1,1}(I)} \le c$, and $\|M_T f^-\|_{W^{1,1}(I)} \le c$.
   e) For every $R > 0$ there exists a constant $c = c(f, R)$ such that for all $T \in (0, R]$, $M_T f^+ \in W^{1,1}(I)$, $M_T f^- \in W^{1,1}(I)$, $\|M_T f^+\|_{W^{1,1}(I)} \le c$, and $\|M_T f^-\|_{W^{1,1}(I)} \le c$.
   If $f : I \to \mathbb{R}$ is absolutely continuous, then a') $f \in W^{1,1}(I)$ is equivalent to b), c), d) and e).

Proof. The implications b) $\to$ e), e) $\to$ d) and d) $\to$ c) are obvious, and a) $\to$ b) is the content of Theorem 2.7. Without loss of generality we may take I to be open, so c) $\to$ a) is a special case of Theorem 3.1. Finally, the last claim follows from the fact that $f \in W^{1,1}(I)$ if and only if f is absolutely continuous and $f \in BV(I)$. $\qquad\square$

Let $f : I \to \mathbb{R}$ be locally integrable. By Theorem 2.7, if $|f| \in BV(I)$ then for every $R > 0$, $M_R f \in W^{1,1}(I)$ boundedly, with bound depending on R. Thus it is natural to ask whether the latter condition alone suffices to ensure that $|f| \in BV(I)$. In other words, we are asking whether the uniform bound condition appearing in parts c), d) and e) of Theorem 3.2 is really needed. The following example shows that the answer is positive.

Example 3.3. *There exists a non-negative function $f \in L^1(\mathbb{R}) \setminus BV(\mathbb{R})$ such that for all $R > 0$, $M_R f \in W^{1,1}(\mathbb{R})$.*

Proof. Let A be the closed set $[-1000, 0] \cup (\cup_{n=0}^\infty [2^{-n}, 2^{-n} + 2^{-n-1}])$, and let f be the upper semicontinuous function χ_A. Fix $R > 0$. Clearly $M_R f \ge f$ everywhere, so by Lemma 3.4 of [AlPe], $M_R f$ is a continuous function. Also, $M_R f|_{\mathbb{R}\setminus(0,2^{-n})}$ is Lipschitz, with $\mathrm{Lip}(M_R f) \le \max\{R^{-1}, 2^{n+1}\}$, by Lemma 3.8 of [AlPe]. Hence, if $E \subset \mathbb{R}$ has measure zero, so does $M_R f(E)$, being a countable union of sets of measure zero. Next we show that $|DM_R f|(\mathbb{R}) < \infty$. Let $n \ge 1$. On intervals of the form $(2^{-n} + 2^{-n-1}, 2^{-n+1})$, if $R > 2^{-n-2}$ then $M_R f > f$, so by Lemma 3.6 of [AlPe] there exists an $x_n \in (2^{-n} + 2^{-n-1}, 2^{-n+1})$ such that $M_R f$ is decreasing on $(2^{-n} + 2^{-n-1}, x_n)$ and increasing on $(x_n, 2^{-n+1})$. Taking this fact into account, it is easy

to see that $V(M_R f, \mathbb{R})$ is decreasing in R, so we may suppose $R \in (0,1)$. Select $N \in \mathbb{N}$ such that $2^{-N+1} < R$. Then for $n > N$,

$$V(M_R f, (2^{-n} + 2^{-n-1}, 2^{-n+1})) = 2\left(1 - M_R f(x_n)\right) \leq 2\left(1 - \frac{R - 2^{-n+1}}{R}\right) \leq \frac{2^{-n+2}}{R}.$$

Hence $|DM_R f|(\mathbb{R}) \leq 2 + 2(N+1) < \infty$. Since $M_R f$ is continuous, of bounded variation, and maps measure zero sets into measure zero sets, by the Banach Zarecki Theorem it is absolutely continuous, so $M_R f \in W^{1,1}(\mathbb{R})$.   □

Of course, using $\mathbb{R}$ above is not necessary, the example can be easily adapted to any other interval I.

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